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On Darboux Lines on Surfaces.

BY JAMES G. HARDY.

1. The problem which forms the subject of this note was first proposed and discussed by M. Darboux in an article entitled "Des courbes tracées sur une surface, et dont la sphère osculatrice est tangente en chaque point à la surface," which appeared in the Comptes Rendus for 1871, and in which he deduced the differential equation of these curves, which is of the second order, and integrated it in the cases of the quadric surfaces and cyclides. Later in the same year Enneper published in the Göttinger Nachrichten a paper in which the geometric signification of the various terms in the differential equation of these lines was pointed out, and in which he deduced their characteristic property that at any point of the curve the radius of its osculating sphere is equal to the radius of curvature of the normal section of the surface having the same tangent. He also showed that if one of these lines be a line of curvature of the surface on which it lies, the surface is the envelope of a sphere of variable radius whose centre describes an arbitrary skew curve. In 1875 an article by M. Ribacour appeared in the Comptes Rendus, in which the author showed that if we trace on a surface S a curve of the kind considered, then each of the osculating planes of this curve cuts S along a section superosculated by a circle. The last paper to appear on this subject is due to M. Cosserat, and is found in the Comptes Rendus for 1895. In it he applies to the investigation of these lines methods analogous to those used in the investigation of geodesics, employing the theory of integrals of determinate form treated for geodesics in Livre VII of the "Leçons" of M. Darboux. I have called these lines Darboux lines because first defined and treated by M. Darboux; M. Cosserat named them D-lines, evidently for the same reasons.

2. Suppose that at any point P of a Darboux line D on a surface we construct a trihedron T_0 having its vertex at P and the normal to the surface as its

axis of z; let ω be the angle which the tangent to D makes with the axis of x of T_0, and let ρ and R be the radius of the osculating sphere and the radius of curvature of D at P respectively. Since by definition the osculating sphere of D is tangent to the surface, its center will be at the intersection of the normal to the surface and the polar line at P. Then, by considering the right triangle whose vertices are the point P, the center of curvature K, and the center of the osculating sphere, $M(x_0 y_0 z_0)$, and applying Meusnier's theorem, we have: the radius of curvature R_n of the normal section of the surface having the same tangent as D, is equal to the radius of the osculating sphere of D at the point considered. Enneper, Gött. Nach., 1871.

Denote by T the radius of torsion of D, by ϕ the angle between the principal normal to D and the normal to the surface at P, and use the formulæ given by Darboux (Théorie générale, II, 359) for the coordinates x_0, y_0, z_0 of the center of the osculating sphere of any line traced on a surface. We have

$$z_0 = R \cos\phi - T\frac{dR}{ds}\sin\phi = 0,$$

from which

$$\tan\phi = -T\frac{1}{R}\,\frac{dR}{ds} \quad \text{and} \quad T = -\frac{\tan\phi}{\dfrac{d}{ds}\log R}.$$

If in the last equation we make $R = \text{const.}$, we will have $T = \infty$ unless $\phi = 0$. Hence *when the radius of curvature of a Darboux line is constant, the line is either plane or geodesic.*

Conversely, when $\phi = 0$ we must have $R = \text{const.}$, since $T \neq 0$; that is, *when a Darboux line is geodesic its radius of curvature is constant.*

If we use the formula

$$\text{geodesic torsion} \equiv \frac{1}{T_g} = \left[\frac{1}{R_1} - \frac{1}{R_2}\right]\sin\omega\cos\omega = \frac{1}{T} - \frac{d\phi}{ds},$$

we find for the geodesic torsion of D

$$\frac{1}{T_g} = -\frac{1}{\tan\phi}\,\frac{d}{ds}\log R_n.$$

Then for $\phi \neq 0$, $\frac{1}{T_g}$ can only be zero when R_n is constant. But when $\frac{1}{T_g} = 0$ the line D is a line of curvature; hence if a line of curvature be a Darboux line, the principal radius of curvature corresponding will be constant. Now in this case one sheaf of the evolute of the surface reduces to a line and, as Monge proved, the surface may be determined as the envelope of a sphere of variable radius whose center describes an arbitrary skew curve. (Enneper, Gött. Nach. l. c.) This same result can be obtained in a different way, as will be seen later. If both systems of lines of curvature be Darboux lines the surface will be a cyclide of Dupin; for M. Bonnet has demonstrated (Journal de l'Ecole Polytechnique, XLII) that if a surface be such that along each line of curvature the principal radius corresponding to that line be constant, the surface is a cyclide of Dupin.

In a note published in the Bulletin des Sciences Mathématiques, 21, 1898, M. Demartres gives some formulæ concerning skew curves which have interesting forms when applied to Darboux lines. The distance between the centers of the osculating spheres at the extremities of an arc SS' of such a line is

$$\delta = \frac{R_n}{T}\,\frac{dR_n}{dR}\,ds;$$

and the angle ψ under which these spheres cut is

$$\psi = \frac{ds}{T}\,\frac{d\log R_n}{d\log R}\,.$$

Along a line of curvature we have $\psi = 0$, but (unless the Darboux line be plane) this requires that R_n be constant, or that $\delta = 0$, in case $R \neq$ constant. That is, if a line of curvature be a Darboux line it is a spherical line or a line of constant curvature.

Using the definitions of absolute and relative spherical torsion given by M. Demartres, we may define Darboux lines as those lines whose absolute spherical torsion is equal to their relative spherical torsion.

The formula just used for geodesic torsion leads to a neat form in the case of Darboux lines on an ellipsoid referred to its lines of curvature. Employing the values given later for $\sin\omega$, $\cos\omega$, ds^2 and $\left(\frac{du}{dv}\right)^2$, we get

$$\frac{1}{T_g} = c\,.\,(u - v)\sqrt{\frac{(u-\varkappa)(v-\varkappa)}{U\quad V}}\,. \qquad c = \text{a constant}$$

This puts into evidence the property of lines of curvature made use of.

The line joining the points P and M generates a ruled surface Σ. The locus of the centers of the osculating spheres of D is the edge of regression of the polar surface, and its osculating plane, which is the normal plane of D, remains tangent to Σ. Therefore the locus of the centers of the osculating spheres of D is an asymptotic line on Σ. Then along any Darboux line the normals to the surface generate a ruled surface on which we know an asymptotic line.

3. The general differential equation of Darboux lines, as given by M. Darboux, is

$$\text{(1)} \qquad \frac{3d^2s}{ds} = \frac{2\Sigma d^2x \,.\, d\dfrac{\partial F}{\partial x} + \Sigma dx \,.\, d^2\dfrac{\partial F}{\partial x}}{\Sigma\, dx \,.\, d\dfrac{\partial F}{\partial x}},$$

where $F = 0$ is the equation of the surface on which the lines lie, and the sign Σ indicates summation with regard to the three coordinates x, y, z. This equation in the case of surfaces of the 2nd degree takes the form

$$\text{(2)} \qquad \varkappa \,.\, ds^2 = \Sigma\left[dx \,.\, d\frac{\partial F}{\partial x}\right], \qquad \varkappa = \text{a constant}.$$

or, introducing elliptic coordinates,

$$\text{(3)} \qquad \frac{(u-\varkappa)\,du^2}{(a^2-u)(b^2-u)(c^2-u)} = \frac{(v-\varkappa)\,dv^2}{(a^2-v)(b^2-v)(c^2-v)}.$$

If we denote by R_n the radius of curvature of a normal section of an ellipsoid, and if the lines of curvature are taken as coordinate lines, R_n will be given by

$$\text{(4)} \qquad R_n = \frac{E\lambda^2 + G}{D\lambda^2 + D''}. \qquad \lambda = \frac{du}{dv}.$$

Let the plane of the normal section considered be passed through the tangent to a Darboux line; calculate λ^2 from (3), and remember that (Darboux, Théorie Générale, II, 379)

$$D = -\frac{\sqrt{abc}\,(u-v)^2}{f(u)\sqrt{-f(u)f(v)}}; \quad D'' = \frac{\sqrt{abc}\,(u-v)^2}{f(v)\sqrt{-f(u)f(v)}};$$

then

$$\text{(5)} \qquad R_n = -\varkappa\frac{\sqrt[4]{R_1R_2}}{\sqrt[4]{abc}},$$

where R_1 and R_2 are the principal radii of curvature at the point considered. We have here a result stated by Enneper: The radius of curvature of the normal section of an ellipsoid having the same tangent as a Darboux line is proportional to the 4th root of the product of the principal radii of curvature at the point considered.

From this result follows another still more interesting. If we calculate the distance from the origin to the tangent plane at any point of an ellipsoid, we find

$$\frac{1}{\delta}=\frac{\sqrt[4]{R_1R_2}}{\sqrt[4]{abc}}. \tag{6}$$

Taking account of (5) and (6), and remembering that for Darboux lines the radius of curvature of the normal section through the tangent is equal to the radius of the osculating sphere, we have

$$\rho=-\varkappa\frac{1}{\delta};$$

that is, the radius ρ of the osculating sphere is inversely proportional to the distance from the center of the ellipsoid to the tangent plane. A better way of stating this same fact is: *the product of the radius of the osculating sphere and the distance from the origin to the tangent plane is constant.*

$$\rho\delta=\text{const.} \tag{7}$$

If the Darboux line be a polhodie, $\delta=$ constant, and hence $\rho=$ constant. The line is therefore spherical, and also (Théorie Générale, II, 381) each normal section tangent to the Darboux line at one of its points is superosculated at that point by a circle.

For the paraboloids $F=\frac{2Xx}{A}+\frac{2Yy}{B}-Z-z=0$, we find the distance δ to be

$$\delta=\frac{-z}{1+\frac{4x^2}{A^2}+\frac{4y^2}{B^2}}.$$

Equation (7) then takes the form

$$\rho\delta=\text{constant}\,.\,z.$$

4. Equation (2) has a very interesting geometric signification, communicated to me for the case of the ellipsoid by Mr. Pell, and which I believe is new. Suppose the surface of the 2nd degree considered be of the form

$$F(x,y,z)=\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}-1=0,$$

then (2) gives

$$\varkappa = \left[\left(\frac{dx}{ds}\right)^2 \frac{1}{a^2} + \left(\frac{dy}{ds}\right)^2 \frac{1}{b^2} + \left(\frac{dz}{ds}\right)^2 \frac{1}{c^2}\right],$$

which, since $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the direction cosines of the tangent at (x, y, z), may be written

$$\varkappa = \left[\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}\right]. \tag{8}$$

If we denote by d the semidiameter whose direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, the coordinates of its end points are $d.\cos\alpha$, $d.\cos\beta$, $d.\cos\gamma$; and, since these coordinates satisfy the equation of the surface,

$$d^2\left[\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}\right] = 1. \tag{9}$$

(8) and (9) combined give

$$d = \text{constant}.$$

That is, *if at any point of a Darboux line of* $F = 0$ *a tangent be drawn, and if we construct the semidiameter parallel to this tangent, the length of this semidiameter is constant along the considered Darboux line.*

5. The different Darboux lines on surfaces of the 2nd degree are obtained by giving different values to the constant $\varkappa$ which enters (3). Suppose in this equation we make $\varkappa = 0$, then

$$\frac{u\,du^2}{U} - \frac{v\,dv^2}{V} = 0,$$

where $U = (a^2 - u)(b^2 - u)(c^2 - u)$ and $V = (a^2 - v)(b^2 - v)(c^2 - v)$. But we have

$$ds^2 = \frac{u - v}{4}\left[\frac{u\,du^2}{U} - \frac{v\,dv^2}{V}\right];$$

hence

$$ds^2 = 0.$$

That is, when $\varkappa = 0$ the Darboux lines are lines of length zero on the surface.

If in (3), after extracting the square root, we put $\varkappa = \infty$, we find

$$\frac{du}{\sqrt{U}} = \frac{dv}{\sqrt{V}}, \tag{10}$$

which is the equation of the right lines on the surface. This may be verified as follows: the equations of the right lines on a quadric are

$$\frac{x}{a} = \frac{z}{c}\, i \,.\cos\phi \pm \sin\phi, \quad \frac{y}{b} = \frac{z}{c}\, i \,.\sin\phi \pm \cos\phi.$$

From these

$$\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2} = 0,$$

which, on substituting for dx, dy, dz their values in terms of du and dv, becomes (10).

6. Consider now an ellipsoid with three unequal axes $a > b > c$, and let us determine for what values of $\varkappa$ the Darboux lines will be real. The elliptic coordinates which enter our equations symmetrically may be distinguished by taking

$$c^2 < u < b^2; \qquad b^2 < v < a^2.$$

Under these conditions $\frac{1}{U}$ is negative and $\frac{1}{V}$ is positive, and consequently in order that the differential equation

$$du \sqrt{\frac{u-\varkappa}{U}} = dv \sqrt{\frac{v-\varkappa}{V}}$$

of the Darboux lines be real, we must have

$$u - \varkappa < 0, \qquad v - \varkappa > 0.$$

That is, the constant $\varkappa$ must satisfy the inequalities

$$c^2 < \varkappa < a^2. \tag{11}$$

That the lines of length zero and the right lines on an ellipsoid are imaginary would appear from (11) if it were not otherwise obvious.

According as the constant $\varkappa$ has different values the following cases may arise:

1. $b^2 > \varkappa > c^2$,
2. $b^2 < \varkappa < a^2$,
3. $\varkappa^2 = a^2$, or b^2, or c^2.

Now if ω be the angle made by the tangent to the considered Darboux line and the axis of x of a trihedron constructed as in §1, we have

$$\cos\omega = \sqrt{E}\,\frac{du}{ds}, \quad \sin\omega = \sqrt{G}\,\frac{dv}{ds}, \quad \tan\omega = \sqrt{\frac{G}{E}}\,\frac{dv}{du};$$

determining $\frac{dv}{du}$ from (3), we find

$$\tan\omega = -\frac{v}{u}\sqrt{\frac{U(u-\varkappa)}{V(v-\varkappa)}}. \tag{12}$$

CASE (1). $\varkappa$ has a u-value $\varkappa = u$, which, substituted in (12), gives

$$\tan\omega = 0, \quad \omega = 0.$$

That is, the Darboux line $\varkappa = u$ is perpendicular to the line of curvature $u = \varkappa$.

CASE (2). When $\varkappa$ has a v-value $\varkappa = v$, equation (12) gives

$$\tan\omega = \infty, \quad \omega = 90^\circ.$$

That is, the Darboux line $\varkappa = v$ is perpendicular to the line of curvature $v = \varkappa$.

CASE (3). To fix the ideas, take $\varkappa = b^2$. The differential equation of the Darboux lines becomes

$$(a^2 - v)(c^2 - v)\,du^2 = (a^2 - u)(c^2 - u)\,dv^2,$$

which is the differential equation of the real circular sections on the ellipsoid. For $\varkappa^2 = a^2$ or c^2 we get the other two systems of circular sections, imaginary of course. (Darboux, C. R., 1871.)

7. In the differential equation of the Darboux lines if we consider u and v as the coordinates of a given point M of the curve, the equation for a given value of $\varkappa$ determines the direction of the curve at M. Conversely, if the direction be given, the equation determines the value of the parameter $\varkappa$. Write

$$ds^2 = \frac{u-v}{4}\left[\frac{u\,du^2}{U} - \frac{v\,dv^2}{V}\right] = \frac{u-v}{4}\left[uP^2 - vQ^2\right];$$

then, calling ϕ the inclination of the Darboux line to the curve $v = \text{constant}$, we have

$$\tan\phi = h\,\frac{Q}{P}, \quad \text{where } h \equiv \sqrt{\frac{u}{v}}. \tag{13}$$

From the equation of the Darboux lines

$$\frac{Q}{P} = \sqrt{\frac{u - \varkappa}{v - \varkappa}}. \tag{14}$$

Combining (13) and (14), we find

$$\frac{v \tan^2 \phi}{h^2} - u = \varkappa \left[\frac{\tan^2 \phi}{h^2} - 1\right].$$

Suppose that for the circular sections we write $\phi \equiv \lambda$, then since $\varkappa = 0$,

$$\frac{v \tan^2 \lambda}{h^2} = u,$$

and

$$\varkappa = \frac{v \left[\tan^2 \phi - \tan^2 \lambda\right]}{\tan^2 \phi - h^2}. \tag{15}$$

When $\phi = 0$ or $90°$, $\varkappa = u$ or v, as should be. Formula (15) enables us to calculate $\varkappa$ for any given point and any given direction.

8. The results of §2 can be arrived at in the following manner: If the coordinates x, y, z of the surface on which the Darboux lines lie be given in terms of two parameters u and v, equation (1) takes a very long form, but one which simplifies when the u and v are taken as the parameters of the lines of curvature on the surface. In this latter case we have

$$\begin{aligned} &3\left[E - \rho D\right] du\, d^2u + 3\left[G - D''\rho\right] dv\, d^2v \\ &= \left\{\rho \left[\frac{\partial D}{\partial u} + \tfrac{1}{2} \frac{D}{E} \; \frac{\partial E}{\partial v}\right] - \tfrac{3}{2} \frac{\partial E}{\partial u}\right\} du^3 \\ &+ \left\{\rho \left[\frac{\partial D}{\partial v} - \tfrac{1}{2} \frac{\partial E}{\partial v} \left(\frac{D''}{G} - 2\,\frac{D}{E}\right)\right] - \tfrac{3}{2} \frac{\partial E}{\partial v}\right\} du^2\, dv \\ &+ \left\{\rho \left[\frac{\partial D''}{\partial u} - \tfrac{1}{2} \frac{\partial G}{\partial u} \left(\frac{D}{E} - 2\,\frac{D''}{G}\right)\right] - \tfrac{3}{2} \frac{\partial G}{\partial u}\right\} du\, d^2v \\ &+ \left\{\rho \left[\frac{\partial D''}{\partial v} + \tfrac{1}{2} \frac{D''}{G} \; \frac{\partial G}{\partial v}\right] - \tfrac{3}{2} \frac{\partial G}{\partial v}\right\} dv^3, \end{aligned} \tag{16}$$

the symmetry of which is remarkable. If the lines $v = \text{const.}$ satisfy (16), we must have

$$\frac{\partial D}{\partial u} - \frac{D}{E}\,\frac{\partial E}{\partial u} = \frac{\partial \frac{1}{\rho_1}}{\partial u} = 0,$$

where ρ_1 is one of the principal radii of curvature at the point considered. Then

$$\rho_1 = f_1(v).$$

Similarly, when the lines $u = \text{const.}$ satisfy (16) we have

$$\rho_2 = f_2(u).$$

Reasoning just as in §2, we find again the result there obtained. Most of the results obtained by Darboux and Enneper follow immediately when the equation of the Darboux lines is expressed in terms of u, v coordinates.

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